

A new random mapping model

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February 8, 2008

Abstract

In this paper we introduce a new random mapping model, $T_n^{\hat{D}}$, which maps the set $\{1, 2, \dots, n\}$ into itself. The random mapping $T_n^{\hat{D}}$ is constructed using a collection of exchangeable random variables $\hat{D}_1, \dots, \hat{D}_n$ which satisfy $\sum_{i=1}^n \hat{D}_i = n$. In the random digraph, $G_n^{\hat{D}}$, which represents the mapping $T_n^{\hat{D}}$, the in-degree sequence for the vertices is given by the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$, and, in some sense, $G_n^{\hat{D}}$ can be viewed as an analogue of the general independent degree models from random graph theory. We show that the distribution of the number of cyclic points, the number of components, and the size of a typical component can be expressed in terms of expectations of various functions of $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$. We also consider two special examples of $T_n^{\hat{D}}$ which correspond to random mappings with preferential and anti-preferential attachment, respectively, and determine, for these examples, exact and asymptotic distributions for the statistics mentioned above. Results for the distribution of the number of successors and predecessors of a typical vertex in $G_n^{\hat{D}}$ in terms of expectations of various functions of $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ are obtained in a companion paper [23].

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‡J.Jaworski acknowledges the generous support by the Marie Curie Intra-European Fellowship No. 501863 (RANDIGRAPH) within the 6th European Community Framework Programme.

1 Introduction

The study of random mapping models was initiated independently by several authors in the 1950s (see [8, 18, 19, 25, 31, 40]) and the properties of these models have received much attention in the literature. In particular, these models have been useful as models for epidemic processes, and have natural applications in cryptology (see, for example, [9, 10, 11, 15, 17, 20, 29, 30, 33, 34, 37, 38, 42]). To date, the most widely studied models have been special cases of a general model denoted by $T_{\mathbf{p}(n)}$, which can be defined as follows: Let $[n]$ denote the set of integers $\{1, 2, \dots, n\}$ and let \mathcal{M}_n denote the set of all mappings from $[n]$ into $[n]$. For each $n \geq 1$, let $\mathbf{p}(n) = \{p_{ij}(n) : 1 \leq i, j \leq n\}$ be an array such that $p_{ij}(n) \geq 0$ for $1 \leq i, j \leq n$ and $\sum_{j=1}^n p_{ij}(n) = 1$ for every $1 \leq i \leq n$, and let $X_1^n, X_2^n, \dots, X_n^n$ be independent random variables such that $\Pr\{X_i^n = j\} = p_{ij}(n)$ for all $1 \leq i, j \leq n$. Then the random mapping $T_{\mathbf{p}(n)} : [n] \rightarrow [n]$ is defined (in terms of the variables $X_1^n, X_2^n, \dots, X_n^n$) by

$$T_{\mathbf{p}(n)}(i) = j \quad \text{iff} \quad X_i^n = j \quad (1.1)$$

for all $1 \leq i, j \leq n$. It follows from (1.1) that the distribution of $T_{\mathbf{p}(n)}$ is given by

$$\Pr\{T_{\mathbf{p}(n)} = f\} = \prod_{i=1}^n p_{if(i)}(n) \quad (1.2)$$

for each $f \in \mathcal{M}_n$. Any mapping $f \in \mathcal{M}_n$ can be represented as a directed graph $G(f)$ on a set of vertices labelled $1, 2, \dots, n$, such that there is a directed edge from vertex i to vertex j in $G(f)$ if and only if $f(i) = j$. So $G_{\mathbf{p}(n)} \equiv G(T_{\mathbf{p}(n)})$ is a random directed graph on a set of vertices labelled $1, 2, \dots, n$ which represents the action of the random mapping $T_{\mathbf{p}(n)}$ on $[n]$. We note that since each vertex in $G_{\mathbf{p}(n)}$ has out-degree 1, the components of $G_{\mathbf{p}(n)}$ consist of directed cycles with directed trees attached. Also, it follows from the definition of $T_{\mathbf{p}(n)}$ that the variables $X_1^n, X_2^n, \dots, X_n^n$ can be interpreted as the independent ‘choices’ of the vertices $1, 2, \dots, n$ in the random digraph $G_{\mathbf{p}(n)}$ (see, in addition, Mutafovich [35] and Jaworski [27]).

The example of $T_{\mathbf{p}(n)}$ which is best understood is the uniform random mapping, $T_n \equiv T_{\mathbf{p}(n)}$, where $p_{ij}(n) = \frac{1}{n}$ for all $1 \leq i, j \leq n$. Much is known (see for example the monograph by Kolchin [32]) about the component structure of the random digraph $G_n \equiv G(T_n)$ which represents T_n . Aldous [1] has shown that the joint distribution of the normalized order statistics for the component sizes in G_n converges to the *Poisson-Dirichlet* (1/2) distribution

on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \text{ for every } i \geq 1\}$. Also, if M_k denotes the number of components of size k in G_n then the joint distribution of (M_1, M_2, \dots, M_b) is close, in the sense of total variation, to the joint distribution of a sequence of independent Poisson random variables when $b = o(n/\log n)$ (see Arratia et.al. [5], [6]) and from this result one obtains a functional central limit theorem for the component sizes (see also [21]). The asymptotic distributions of variables such as the number of predecessors and the number of successors of a vertex in G_n are also known (see [10, 11, 15, 29, 30, 33, 34, 37]). In another direction, Berg, Jaworski, and Mutařchiev (see [12, 27, 28, 29, 30]) have investigated the structure of $G_{\mathbf{p}(n)}$ when $\mathbf{p}(n)$ is given by $p_{ii}(n) = q$ for some $0 \leq q \leq 1$ and all $1 \leq i \leq n$, and $p_{ij}(n) = \frac{1-q}{n-1}$ for all $1 \leq i, j \leq n$ such that $i \neq j$. Finally, Aldous, Miermont, and Pitman (see [2] and [3]) have recently investigated the asymptotic structure of $G_{\mathbf{p}(n)}$, where $\mathbf{p}(n)$ is given by $p_{ij}(n) = p_j(n) > 0$ for all $1 \leq i, j \leq n$, by using an ingenious coding of the mapping $T_{\mathbf{p}(n)}$ as a stochastic process on the interval $[0, 1]$. Their results are closely related to earlier work on the relationship between random mappings and random forests (see Pitman [36] and references therein).

The common feature in all the models discussed above is that each vertex in $G_{\mathbf{p}(n)}$ ‘chooses’ the vertex that it is mapped to independently of the ‘choices’ made by all other vertices. In this paper we introduce a new random mapping model in which the vertex ‘choices’ are not necessarily independent. The definition of the model is motivated, in part, by developments in the general theory of random graphs. In recent years models for random graphs with a specified degree sequence have received much attention as models for complex networks such as the internet. Loosely speaking, such a random graph on n labelled vertices can be constructed by starting with a collection of i.i.d., non-negative, integer-valued random variables D_1, D_2, \dots, D_n and adding edges, at random, to the graph until each vertex i has degree D_i in the constructed random graph. Of interest in such models is the relationship between the component structure of the random graph and the distribution of the variables D_1, D_2, \dots, D_n . In another direction random graph models with ‘preferential attachment’ have been constructed in order to model the *evolving* structure of complex networks. In such models edges are added sequentially to the graph and new edges are more likely to be attached to vertices that already have relatively high degree in the evolving graph. The literature on these new developments in the theory of random graphs is extensive, but a good bibliography is provided by Bonato’s survey paper [14].

In this paper we consider a random mapping analogue for both the independent degree models and the preferential attachment models described above. We show that in the case of random mappings these analogues are equivalent and we develop a calculus for determining the distributions of various important random mapping statistics which is based on the underlying distribution of the in-degrees of the vertices in the directed graph which represents the random mapping. The paper is organized as follows. In Section 2 we define carefully the new random mapping model. In Section 3 we derive the calculus for this new model. In Section 4 we define both a random mapping model with preferential attachment and with anti-preferential attachment. We show that both of these models are equivalent, under certain distribution assumptions, to the special examples of the general model defined in Section 2. For these special examples we investigate the distribution of the number of cyclic points, the distribution of the number of components, the probability of connectedness, and the distribution of the size of a typical component.

2 The model

In order to define our new random mapping model, we adopt the following notation. For $n \geq 1$, \mathcal{M}_n denotes the set of all mappings $f : [n] \rightarrow [n]$, where $[n] \equiv \{1, 2, \dots, n\}$, and $G(f)$ denotes, as described in the Introduction, the directed graph on n labelled vertices which represents the mapping $f \in \mathcal{M}_n$. In addition, for $1 \leq i \leq n$, $d_i(f)$ denotes the in-degree of vertex i in the digraph $G(f)$, and we let $\vec{d}(f) \equiv (d_1(f), \dots, d_n(f))$.

Now suppose that $\hat{D}_1, \dots, \hat{D}_n$ is a collection of exchangeable random variables such that $\sum_{i=1}^n \hat{D}_i = n$, then we construct a probability measure $\mathbf{P}_n^{\hat{D}}$ on \mathcal{M}_n as follows. For $f \in \mathcal{M}_n$, define

$$\mathbf{P}_n^{\hat{D}}\{f\} = \frac{\prod_{i=1}^n (d_i(f))!}{n!} \Pr\{\hat{D}_i = d_i(f), 1 \leq i \leq n\}. \quad (2.1)$$

It is clear from the definition of $\mathbf{P}_n^{\hat{D}}$ that for any $f, g \in \mathcal{M}_n$ such that $\vec{d}(f) = \vec{d}(g)$, we have $\mathbf{P}_n^{\hat{D}}\{f\} = \mathbf{P}_n^{\hat{D}}\{g\}$. Given the probability measure $\mathbf{P}_n^{\hat{D}}$, we can define the random mapping $T_n^{\hat{D}}$ which takes values in \mathcal{M}_n and has distribution given by

$$\Pr\{T_n^{\hat{D}} = f\} = \mathbf{P}_n^{\hat{D}}\{f\} \quad (2.2)$$

for every $f \in \mathcal{M}_n$, and we let $G_n^{\hat{D}} \equiv G(T_n^{\hat{D}})$ denote the random digraph on n labelled vertices which represents $T_n^{\hat{D}}$.

An important class of examples can be constructed as follows. Let D_1, D_2, \dots, D_n be i.i.d. non-negative integer-valued random variables. It does not make sense to construct directly a random mapping digraph with in-degrees given by D_1, D_2, \dots, D_n since the sum of the vertex in-degrees in a random mapping digraph on n vertices always equals n . So, instead, we let $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ be a sequence of random variables such that their joint distribution is given by

$$\Pr\{\hat{D}_i = d_i, 1 \leq i \leq n\} = \Pr\left\{D_i = d_i, 1 \leq i \leq n \mid \sum_{i=1}^n D_i = n\right\}$$

and we use $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ to construct $T_n^{\hat{D}}$ and $G_n^{\hat{D}}$. This gives us the natural analogue of the general i.i.d. degree model discussed in the Introduction. We also remark that it is easy to check that if D_1, D_2, \dots, D_n are i.i.d. *Poisson*(1) variables, then the corresponding random mapping $T_n^{\hat{D}}$ is just the usual uniform random mapping. We will see in Section 4 below that there are interesting interpretations $T_n^{\hat{D}}$ in the cases where the underlying i.i.d. variables D_1, D_2, \dots, D_n have (i) a generalised negative binomial distribution, and (ii) a *Bin*($m, 1/m$) distribution for some $m \geq 2$.

3 Results

In this section we develop a calculus, in terms of the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$, for determining the distributions of various random variables associated with the structure of $G_n^{\hat{D}}$. The first variable we consider is the number of cyclic vertices in the random digraph $G_n^{\hat{D}}$.

A vertex $i \in [n]$ is a cyclic vertex for the mapping $f \in \mathcal{M}_n$ (and for the corresponding digraph $G(f)$) if there is some $k \geq 1$ such that $f^{(k)}(i) = i$, where $f^{(k)}$ is the k^{th} iterate of the function f . We define $X_n(f)$ to be the number of cyclic vertices of $f \in \mathcal{M}_n$ and we let $X_n^{\hat{D}} \equiv X_n(T_n^{\hat{D}})$ denote the number of cyclic vertices in $G_n^{\hat{D}}$. Then we have

Theorem 1. *For $1 \leq k \leq n-1$*

$$\Pr\{X_n^{\hat{D}} = k\} = \frac{k}{n-k} E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k)$$

and

$$\Pr\{X_n^{\hat{D}} = n\} = \Pr\{\hat{D}_i = 1, 1 \leq i \leq n\}.$$

Proof. (of Theorem 1) We begin by considering the case $1 \leq k \leq n-1$. For $f \in \mathcal{M}_n$, let $\mathcal{L}_n(f)$ denote the set of cyclic vertices for the mapping f , and let $\mathcal{L}_n^{\hat{D}} \equiv \mathcal{L}_n(T_n^{\hat{D}})$. Then we have

$$\Pr\{X_n^{\hat{D}} = k\} = \sum_{L \subseteq [n] \text{ s.t. } |L|=k} \Pr\{\mathcal{L}_n^{\hat{D}} = L\}.$$

We fix $L = \{1, 2, \dots, k\} = [k]$ and observe that

$$\begin{aligned} \Pr\{\mathcal{L}_n^{\hat{D}} = [k]\} = \\ \sum_{\substack{\{d_i\} \\ \text{s.t. } \sum d_i = n}} \Pr\left\{\mathcal{L}_n^{\hat{D}} = [k] \mid \hat{D}_i = d_i, 1 \leq i \leq n\right\} \Pr\{\hat{D}_i = d_i, 1 \leq i \leq n\}. \end{aligned}$$

Now fix $\vec{d} = (d_1, d_2, \dots, d_n)$ such that $\sum_{i=1}^n d_i = n$ and let

$$\mathcal{M}_n(\vec{d}) = \{f \in \mathcal{M}_n : d_i(f) = d_i, 1 \leq i \leq n\}.$$

Then it follows from (2.1) and (2.2) that

$$\Pr\left\{\mathcal{L}_n^{\hat{D}} = [k] \mid \hat{D}_i = d_i, 1 \leq i \leq n\right\} = \frac{|\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\}|}{n! (\prod_{i=1}^n d_i!)^{-1}}.$$

In order to count the set $\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\}$, we first construct a bijection between the set, $\mathcal{S}_{\vec{d}}$, of sequences $\mathbf{x} = (x_1, x_2, \dots, x_n)$ such that for any $\mathbf{x} \in \mathcal{S}_{\vec{d}}$ and $1 \leq i \leq n$, we have $|\{m : x_m = i\}| = d_i$ and the set of mappings $\mathcal{M}_n(\vec{d})$.

The bijection is defined in terms of an algorithm which, for any sequence $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{S}_{\vec{d}}$, constructs, after a series of ‘rounds’, a corresponding mapping $f_{\mathbf{x}} \in \mathcal{M}(\vec{d})$. Informally, the algorithm works as follows. In the first round the vertices $i \in [n]$ with $d_i = 0$ are mapped (in a way which is determined by the sequence \mathbf{x}) to vertices in $[n]$ which have in-degree greater than 0. After the first round the ‘availability’ of some vertices to receive directed edges in the corresponding directed graph will have been reduced because the algorithm has mapped some vertices to them. A vertex i becomes

‘unavailable’ as soon as it is ‘full’, i.e. d_i vertices have been mapped to it. So, at the beginning of the second round some vertices which were ‘available’ in the first round may be ‘unavailable’ at the start of the second round. These new ‘unavailable’ vertices are mapped to ‘available’ vertices in the second round and this construction continues until the mapping $f_{\mathbf{x}}$ is completely defined.

In order to describe rigorously the algorithm which constructs $f_{\mathbf{x}}$, we introduce the following notation. For any vertex $i \in [n]$, any $m \geq 1$, and any $\mathbf{x} \in \mathcal{S}_{\vec{d}}$, we let $a_i(m, \mathbf{x})$ denote the ‘availability’ of vertex i at the start of round m . Initially we set $a_i(1, \mathbf{x}) = d_i$ for all $i \in [n]$. The values of $a_i(m, \mathbf{x})$ when $m \geq 2$ are determined recursively by the algorithm. For $\mathbf{x} \in \mathcal{S}_{\vec{d}}$ and $m \geq 1$ we define the sets

$$\mathcal{Y}_m(\mathbf{x}) = \{i \in [n] : a_i(m, \mathbf{x}) = 0\}, \quad \mathcal{Y}_0(\mathbf{x}) = \emptyset, \quad \mathcal{Z}_m(\mathbf{x}) = \mathcal{Y}_m(\mathbf{x}) \setminus \mathcal{Y}_{m-1}(\mathbf{x}).$$

Finally, for $\mathbf{x} \in \mathcal{S}_{\vec{d}}$, let $y_0(\mathbf{x}) = 0$ and for $m \geq 1$, let $y_m(\mathbf{x}) = |\mathcal{Y}_m(\mathbf{x})|$ and $z_m(\mathbf{x}) = |\mathcal{Z}_m(\mathbf{x})|$.

The algorithm: Given $\mathbf{x} \in \mathcal{S}_{\vec{d}}$, the corresponding mapping $f_{\mathbf{x}} \in \mathcal{M}_n(\vec{d})$ is constructed as follows.

Step 1. Set $m = 1$.

Step 2. If $\mathcal{Z}_m(\mathbf{x}) \neq \emptyset$, then

- list the elements of $\mathcal{Z}_m(\mathbf{x})$ in increasing order: $i_1 < i_2 < \dots < i_{z_m(\mathbf{x})}$. For $1 \leq k \leq z_m(\mathbf{x})$, define $f_{\mathbf{x}}(i_k) = x_{y_{m-1}(\mathbf{x})+k}$, where x_t is the t^{th} term in the sequence $\mathbf{x} = (x_1, x_2, \dots, x_n)$.
- Next, for each $j \in [n]$, set

$$a_j(m+1, \mathbf{x}) = a_j(m, \mathbf{x}) - |\{i \in \mathcal{Z}_m(\mathbf{x}) : f_{\mathbf{x}}(i) = j\}|.$$

- Go to Step 3.

Otherwise, if $\mathcal{Z}_m(\mathbf{x}) = \emptyset$ (i.e. $\mathcal{Y}_{m-1}(\mathbf{x}) = \mathcal{Y}_m(\mathbf{x})$), then

- list the set $[n] \setminus \mathcal{Y}_m(\mathbf{x})$ in increasing order: $i_1 < i_2 < \dots < i_{n-y_m(\mathbf{x})}$. For $1 \leq k \leq n - y_m(\mathbf{x})$, let $f_{\mathbf{x}}(i_k) = x_{y_m(\mathbf{x})+k}$. This completes the construction of the mapping $f_{\mathbf{x}}$.

Step 3. Set $m = m + 1$ and go to Step 2.

We mention here that we have recently learned that Blitzstein and Diaconis [13] have used an algorithm to construct labelled trees with a given degree sequence which is similar in spirit to the algorithm described above. It is clear from the definition of the algorithm, that if $\mathbf{x}, \mathbf{x}' \in \mathcal{S}_{\vec{d}}$ and $\mathbf{x} \neq \mathbf{x}'$, then $f_{\mathbf{x}} \neq f_{\mathbf{x}'}$. In particular, if $k(\mathbf{x}, \mathbf{x}') \equiv \min\{k \in [n] : x_k \neq x'_k\}$ then there is some $i \in [n]$ such that $f_{\mathbf{x}}(i) = x_{k(\mathbf{x}, \mathbf{x}')}(\mathbf{x})$ and $f_{\mathbf{x}'}(i) = x'_{k(\mathbf{x}, \mathbf{x}')}$. Since $|\mathcal{S}_{\vec{d}}| = \frac{n!}{d_1! \dots d_n!} = |\mathcal{M}_n(\vec{d})|$, it follows that the algorithm constructs a bijection between $\mathcal{S}_{\vec{d}}$ and $\mathcal{M}_n(\vec{d})$. Hence, to count $\{f \in \mathcal{M}_n(\vec{d}) : \mathcal{L}_n(f) = [k]\}$ it suffices to count $\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \mathcal{L}_n(f_{\mathbf{x}}) = [k]\}$. To this end, we prove the following lemmas.

Lemma 1. *For each $\mathbf{x} \in \mathcal{S}_{\vec{d}}$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we define $t(\mathbf{x})$ as follows:*

$$t(\mathbf{x}) = \min \left\{ t : |\{x_t, x_{t+1}, \dots, x_n\}| = n - t + 1 \right\}.$$

Then

$$\mathcal{L}_n(f_{\mathbf{x}}) = \{x_{t(\mathbf{x})}, x_{t(\mathbf{x})+1}, \dots, x_n\}.$$

Proof. (of Lemma 1) Suppose that $\mathbf{x} \in \mathcal{S}_{\vec{d}}$ and suppose that the algorithm terminates in round \hat{m} , then the first step is to show that

$$\mathcal{L}_n(f_{\mathbf{x}}) = [n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x}). \quad (3.1)$$

We begin with a few simple observations. First, it follows from the description of the algorithm that at the beginning of the m^{th} round, where $1 \leq m \leq \hat{m}$, the vertices in $[n] \setminus \mathcal{Y}_{m-1}(\mathbf{x})$ have not yet been assigned their image under the mapping $f_{\mathbf{x}}$. Also, for every $i \in [n]$, we have

$$a_i(m, \mathbf{x}) = |\{k : x_k = i, y_{m-1}(\mathbf{x}) + 1 \leq k \leq n\}|. \quad (3.2)$$

In other words, $a_i(m, \mathbf{x}) = r$ if and only if i appears r times in the sequence $(x_{y_{m-1}(\mathbf{x})+1}, x_{y_{m-1}(\mathbf{x})+2}, \dots, x_n)$. In addition, it follows from (3.2) and from the definition of $\mathcal{Y}_m(\mathbf{x})$, that

$$n - y_{m-1}(\mathbf{x}) = \sum_{i \in [n]} a_i(m, \mathbf{x}) = \sum_{i \in [n] \setminus \mathcal{Y}_m(\mathbf{x})} a_i(m, \mathbf{x})$$

since $a_i(m, \mathbf{x}) = 0$ for every $i \in \mathcal{Y}_m(\mathbf{x})$. Now if $m = \hat{m}$, then $\mathcal{Y}_{\hat{m}}(\mathbf{x}) = \mathcal{Y}_{\hat{m}-1}(\mathbf{x})$ and so

$$n - y_{\hat{m}}(\mathbf{x}) = n - y_{\hat{m}-1}(\mathbf{x}) = \sum_{i \in [n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x})} a_i(\hat{m}, \mathbf{x}).$$

This implies immediately that

$$a_i(\hat{m}, \mathbf{x}) = 1 \text{ for all } i \in [n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x}). \quad (3.3)$$

Hence the last step of the algorithm generates a permutation on the set $[n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x})$ i.e.,

$$\{x_{y_{\hat{m}}(\mathbf{x})+1}, \dots, x_n\} = [n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x}) \subseteq \mathcal{L}_n(f_{\mathbf{x}}). \quad (3.4)$$

Suppose now that $i \in \mathcal{Y}_{\hat{m}}(\mathbf{x})$ (i.e., $i \in [n]$ and $a_i(m, \mathbf{x}) = 0$ for some $1 \leq m \leq \hat{m} - 1$). Then, since

$$\mathcal{Y}_1(\mathbf{x}) \subseteq \mathcal{Y}_2(\mathbf{x}) \subseteq \dots \subseteq \mathcal{Y}_{\hat{m}-1}(\mathbf{x}) = \mathcal{Y}_{\hat{m}}(\mathbf{x}),$$

we can define

$$m_i = \min\{m : a_i(m, \mathbf{x}) = 0\}.$$

If $m_i = 1$, then $a_i(1, \mathbf{x}) = d_i = 0$, and we must have $i \notin \mathcal{L}_n(f_{\mathbf{x}})$ since vertex i has in-degree 0 in $G_n(f_{\mathbf{x}})$. So, suppose that $2 \leq m_i \leq \hat{m} - 1$ (and $d_i = a_i(1, \mathbf{x}) > 0$). It follows from the description of the algorithm and the definition of m_i , that at the start of round m_i vertex i is ‘unavailable’ and $f_{\mathbf{x}}^{-1}(i) \equiv \{j : f_{\mathbf{x}}(j) = i\} \subseteq \mathcal{Y}_{m_i-1}$. Since the value of $f_{\mathbf{x}}(i)$ is determined during round m_i and since $a_i(m_i, \mathbf{x}) = 0$, it follows that $f_{\mathbf{x}}(i) \neq i$. It is also clear from the description of the algorithm that for *any* $j \in [n] \setminus \mathcal{Y}_{m_i-1}$, we have $f_{\mathbf{x}}(j) \notin \mathcal{Y}_{m_i-1}$ since, for every $k \in \mathcal{Y}_{m_i-1}$ and for every $m \geq m_i$, $a_k(m, \mathbf{x}) = 0$ (i.e., vertex $k \in \mathcal{Y}_{m_i-1}$ is ‘unavailable’ in *every* round $m \geq m_i$). It follows by induction that

$$f_{\mathbf{x}}^{(t)}(i) \notin \mathcal{Y}_{m_i-1} \quad (3.5)$$

for every $t \geq 1$. Now suppose that $i \in \mathcal{L}_n(f_{\mathbf{x}})$, and in particular, that $f_{\mathbf{x}}^{(t')}(i) = i$ for some $t' \geq 2$, then $f_{\mathbf{x}}^{(t'-1)}(i) \in f_{\mathbf{x}}^{-1}(i) \subseteq \mathcal{Y}_{m_i-1}$. But this contradicts (3.5), so we must have $i \notin \mathcal{L}_n(f_{\mathbf{x}})$. Therefore

$$\mathcal{Y}_{\hat{m}}(\mathbf{x}) \subseteq [n] \setminus \mathcal{L}_n(f_{\mathbf{x}}).$$

This together with (3.4) implies (3.1).

Next, we note that (3.3), (3.4), and the definition of $t(\mathbf{x})$, imply

$$\{x_{y_{\hat{m}(\mathbf{x})+1}}, \dots, x_n\} = [n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x}) \subseteq \{x_{t(\mathbf{x})}, \dots, x_n\}.$$

Thus to show that

$$[n] \setminus \mathcal{Y}_{\hat{m}}(\mathbf{x}) = \{x_{t(\mathbf{x})}, \dots, x_n\}, \quad (3.6)$$

it suffices to show that

$$x_{y_{\hat{m}(\mathbf{x})}} \notin \{x_{t(\mathbf{x})}, \dots, x_n\}.$$

Suppose that $x_{y_{\hat{m}(\mathbf{x})}} \in \{x_{t(\mathbf{x})}, \dots, x_n\}$. Then in the $\hat{m} - 1^{st}$ round there is some $j' \in \mathcal{Z}_{\hat{m}-1}$ such that j' is assigned to $x_{y_{\hat{m}(\mathbf{x})}}$. So at the beginning of the $\hat{m} - 1^{st}$ round of the algorithm $x_{y_{\hat{m}(\mathbf{x})}}$ was available, but it is unavailable at the start of the \hat{m}^{th} round (since $x_{y_{\hat{m}(\mathbf{x})}} \notin [n] \setminus \mathcal{Y}_{\hat{m}}$). Since the $\hat{m} - 1^{th}$ round of the algorithm cannot generate *new* unavailable vertices, we have a contradiction. Hence (3.6) holds and Lemma 1 follows immediately from (3.1). \square

It follows from Lemma 1 that

$$\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \mathcal{L}_n(f_{\mathbf{x}}) = [k]\} = \{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, \dots, x_n\} = [k] \text{ and } x_{n-k} \in [k]\}.$$

So routine counting arguments yield

$$\begin{aligned} & \Pr \left\{ \mathcal{L}_n^{\hat{D}} = [k] \middle| \hat{D}_i = d_i, 1 \leq i \leq n \right\} \\ &= \frac{|\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, \dots, x_n\} = [k] \text{ and } x_{n-k} \in [k]\}| \times \prod_{i=1}^n d_i!}{n!} \\ &= \binom{n}{k}^{-1} \left(\frac{1}{n-k} \right) \sum_{i=1}^k d_1 d_2 \cdots d_{i-1} (d_i - 1) d_i d_{i+1} \cdots d_k \end{aligned}$$

for $1 \leq k \leq n - 1$. Hence

$$\Pr \{ \mathcal{L}_n^{\hat{D}} = [k] \} = \binom{n}{k}^{-1} \left(\frac{1}{n-k} \right) \sum_{i=1}^k E(\hat{D}_1 \cdots (\hat{D}_i - 1) \hat{D}_i \cdots \hat{D}_k)$$

$$= \binom{n}{k}^{-1} \frac{k}{n-k} E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k)$$

since $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_k$ are exchangeable. Repeating the argument given above and using the exchangeability of the variables $\hat{D}_1, \dots, \hat{D}_n$, we also obtain

$$\Pr\{\mathcal{L}_n^{\hat{D}} = L\} = \binom{n}{k}^{-1} \frac{k}{n-k} E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k)$$

for any $L \subseteq [n]$ such that $|L| = k$. Summing over all such sets L , we obtain

$$\Pr\{X_n^{\hat{D}} = k\} = \frac{k}{n-k} E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k).$$

Finally, we consider the case $k = n$. Clearly $X_n^{\hat{D}} = n$ if and only if $f_{\mathbf{x}}$ is a permutation of $[n]$, and $f_{\mathbf{x}}$ is a permutation of $[n]$ if and only if $\hat{D}_i = 1$ for all $1 \leq i \leq n$. Hence

$$\Pr\{X_n^{\hat{D}} = n\} = \Pr\{\hat{D}_i = 1, 1 \leq i \leq n\}$$

as required, and the theorem is proved. \square

Let $N_n^{\hat{D}}$ denote the number of components in $G_n^{\hat{D}}$. Let $\sigma(m)$ be a uniform random permutation on an m -element set and let $N_{\sigma(m)}$ denote the number of cycles in the random permutation $\sigma(m)$. Then we have

Corollary 1. *For $1 \leq \ell \leq n$,*

$$\begin{aligned} \Pr\{N_n^{\hat{D}} = \ell\} &= \sum_{k=\ell}^n \frac{|s(k, \ell)|}{(k-1)!(n-k)} E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k) \\ &\quad + \frac{|s(n, \ell)|}{n!} \Pr\{\hat{D}_i = 1, 1 \leq i \leq n\}, \end{aligned}$$

where $s(\cdot, \cdot)$ are the Stirling numbers of the first kind.

Proof. Recall that the mapping $T_n^{\hat{D}}$ restricted to $\mathcal{L}_n^{\hat{D}}$, the cyclic vertices of $T_n^{\hat{D}}$, is a permutation of $\mathcal{L}_n^{\hat{D}}$. The corollary follows from the observation that the number of components in $G_n^{\hat{D}}$ equals the number of cycles in the permutation of $\mathcal{L}_n^{\hat{D}}$ by $T_n^{\hat{D}}$. So $N_n^{\hat{D}} = \ell$ if and only if the mapping $T_n^{\hat{D}}$ restricted to $\mathcal{L}_n^{\hat{D}}$ is a permutation with ℓ cycles.

First suppose that $1 \leq \ell \leq k \leq n-1$, and $L \subseteq [n]$ such that $|L| = k$. Also fix the in-degree sequence $\vec{d} = (d_1, d_2, \dots, d_n)$ such that $\sum_{i=1}^n d_i = n$, and such that $d_i \geq 1$ for every $i \in L$ and $d_{i'} \geq 2$ for *some* $i' \in L$. Recall that for any $f, g \in \mathcal{M}_n(\vec{d})$ we have $\mathbf{P}_n^{\hat{D}}(f) = \mathbf{P}_n^{\hat{D}}(g)$, so it follows from the bijection described in the proof of Theorem 1 that

$$\begin{aligned} & \Pr\left\{N_n^{\hat{D}} = \ell \mid \mathcal{L}_n^{\hat{D}} = L, \hat{D}_i = d_i, 1 \leq i \leq n\right\} = \\ &= \frac{\left|\left\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, \dots, x_n\} = L, x_{n-k} \in L, \text{permutation } f_{\mathbf{x}}|_L \text{ has } \ell \text{ cycles}\right\}\right|}{\left|\left\{\mathbf{x} \in \mathcal{S}_{\vec{d}} : \{x_{n-k+1}, \dots, x_n\} = L, x_{n-k} \in L\right\}\right|} \\ &= \Pr\{N_{\sigma(k)} = \ell\}. \end{aligned}$$

Since the set $L \subseteq [n]$ and the degree sequence \vec{d} were arbitrary, it follows that

$$\Pr\left\{N_n^{\hat{D}} = \ell \mid X_n^{\hat{D}} = k\right\} = \Pr\{N_{\sigma(k)} = \ell\}. \quad (3.7)$$

In the case $X_n^{\hat{D}} = n$, the mapping $T_n^{\hat{D}}$ is a random permutation of $[n]$ (since $\mathcal{L}_n^{\hat{D}} = [n]$), so for $1 \leq \ell \leq n$ we also have

$$\Pr\left\{N_n^{\hat{D}} = \ell \mid X_n^{\hat{D}} = n\right\} = \Pr\{N_{\sigma(n)} = \ell\}. \quad (3.8)$$

It follows that

$$\begin{aligned} \Pr\{N_n^{\hat{D}} = \ell\} &= \sum_{k=\ell}^n \Pr\left\{N_n^{\hat{D}} = \ell \mid X_n^{\hat{D}} = k\right\} \Pr\{X_n^{\hat{D}} = k\} \\ &= \sum_{k=1}^n \Pr\{N_{\sigma(k)} = \ell\} \Pr\{X_n^{\hat{D}} = k\}. \end{aligned}$$

Let $s(\cdot, \cdot)$ denote the Stirling numbers of the first kind, then it is well known that there are $|s(k, l)|$ permutations of k -element set with exactly l cycles, i.e.,

$$\Pr\{N_{\sigma(k)} = \ell\} = \frac{|s(k, \ell)|}{k!},$$

which implies the assertion of the corollary. \square

Let $\mathcal{B}_n^{\hat{D}}$ denote the event that the random graph $G_n^{\hat{D}}$ is connected. Then since $\mathcal{B}_n^{\hat{D}} = \{N_n^{\hat{D}} = 1\}$, we obtain the following result immediately from the proof of Corollary 1:

Corollary 2.

$$\begin{aligned} \Pr\{\mathcal{B}_n^{\hat{D}}\} &= \sum_{k=1}^n \frac{1}{k} \Pr\{X_n^{\hat{D}} = k\} \\ &= \sum_{k=1}^{n-1} \frac{E((\hat{D}_1 - 1)\hat{D}_1\hat{D}_2 \cdots \hat{D}_k)}{n - k} + \frac{\Pr\{\hat{D}_i = 1, 1 \leq i \leq n\}}{n}. \end{aligned}$$

Finally, suppose that ξ_1, ξ_2, \dots is a sequence of independent indicator variables such that, for $k \geq 1$, $\Pr\{\xi_k = 1\} = \frac{1}{k}$ and such that ξ_1, ξ_2, \dots and $X_n^{\hat{D}}$ are independent. It is well known (see [16]) that for $m \geq 1$

$$N_{\sigma(m)} \stackrel{d}{\sim} \sum_{k=1}^m \xi_k, \quad (3.9)$$

and that $(N_{\sigma(m)} - \log m)/\sqrt{\log m}$ converges in distribution to the standard $N(0, 1)$ distribution. It is an easy consequence of (3.7)-(3.9), that

Corollary 3. For $n \geq 1$,

$$N_n^{\hat{D}} \stackrel{d}{\sim} \sum_{k=1}^{X_n^{\hat{D}}} \xi_k. \quad (3.10)$$

We note that (3.10) is useful for investigating the asymptotic distribution of $N_n^{\hat{D}}$.

Next, we consider the distribution of the size of a ‘typical’ component of $G_n^{\hat{D}}$. Let $C_1^{\hat{D}}(n)$ denote the size of the component in $G_n^{\hat{D}}$ which contains the vertex 1, then the distribution of $C_1^{\hat{D}}(n)$ is given by the following theorem.

Theorem 2. For $1 \leq \ell \leq n$, let $D'_1, D'_2, \dots, D'_\ell$ be a sequence of variables with joint distribution given by

$$\Pr\{D'_i = d_i, 1 \leq i \leq \ell\} = \Pr\{\hat{D}_i = d_i, 1 \leq i \leq \ell \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\},$$

then

$$\Pr\{C_1^{\hat{D}}(n) = \ell\} = \frac{\ell}{n} \Pr\{\mathcal{B}_\ell^{D'}\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}.$$

Proof. Fix $1 \leq \ell \leq n$ and let $\mathcal{C}_1^{\hat{D}}(n)$ denote the vertex set of the component of $G_n^{\hat{D}}$ which contains the vertex 1. Then we have

$$\Pr\{\mathcal{C}_1^{\hat{D}}(n) = \ell\} = \sum_{\substack{C \subseteq [n] \text{ s.t. } 1 \in C \\ \text{and } |C| = \ell}} \Pr\{\mathcal{C}_1^{\hat{D}}(n) = C\}.$$

Now fix $C = [\ell]$ and observe that if $\mathcal{C}_1^{\hat{D}}(n) = [\ell]$ then $T_n^{\hat{D}}$ must map $[\ell]$ into $[\ell]$, and, in particular, we must have $\sum_{i=1}^{\ell} \hat{D}_i = \ell$. So

$$\Pr\{\mathcal{C}_1^{\hat{D}}(n) = [\ell]\} = \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}. \quad (3.11)$$

Next, by summing over all degree sequences $\vec{d} = (d_1, d_2, \dots, d_n)$ such that $\sum_{i=1}^{\ell} d_i = \ell$ and $\sum_{i=1}^n d_i = n$, we obtain

$$\begin{aligned} & \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \\ &= \sum_{\substack{\vec{d} \text{ s.t. } \sum_{i=1}^{\ell} d_i = \ell, \\ \sum_{i=1}^n d_i = n}} \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \mid \hat{D}_i = d_i, 1 \leq i \leq n\right\} \\ & \quad \times \Pr\left\{\hat{D}_i = d_i, 1 \leq i \leq n \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}. \end{aligned} \quad (3.12)$$

Now fix $\vec{d} = (d_1, \dots, d_n)$ such that $\sum_{i=1}^{\ell} d_i = \ell$ and $\sum_{i=1}^n d_i = n$, and define $M(d_1, d_2, \dots, d_{\ell})$ to be the number of mappings $f : [\ell] \rightarrow [\ell]$ such that $d_i(f) = d_i$ for $1 \leq i \leq \ell$ and $G_{\ell}(f)$ is connected. Then we have

$$\begin{aligned} \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \mid \hat{D}_i = d_i, 1 \leq i \leq n\right\} &= \frac{M(d_1, \dots, d_{\ell})(n - \ell)!}{d_{\ell+1}! \cdots d_n!} \times \frac{d_1! \cdots d_n!}{n!} \\ &= \binom{n}{\ell}^{-1} M(d_1, \dots, d_{\ell}) \times \frac{d_1! \cdots d_{\ell}!}{\ell!} \\ &= \binom{n}{\ell}^{-1} \Pr\left\{\mathcal{B}_{\ell}^{\mathcal{D}'} \mid D'_i = d_i, 1 \leq i \leq \ell\right\}. \end{aligned} \quad (3.13)$$

Now for any degree sequence $\vec{d} = (d_1, \dots, d_n)$, we define $\vec{d}_\ell = (d_1, \dots, d_\ell)$ and $\vec{d}_\ell' = (d_{\ell+1}, \dots, d_n)$ (so $\vec{d} = (\vec{d}_\ell, \vec{d}_\ell')$). Then, by substituting the RHS of (3.13) into (3.12), we obtain

$$\begin{aligned}
& \Pr\left\{\mathcal{C}_1^{\hat{D}}(n) = [\ell] \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \\
&= \binom{n}{\ell}^{-1} \sum_{\substack{\vec{d}_\ell \text{ s.t.} \\ \sum_{i=1}^{\ell} d_i = \ell}} \sum_{\substack{\vec{d}_\ell' \text{ s.t.} \\ \sum_{i=\ell+1}^n d_i = n - \ell}} \Pr\left\{\mathcal{B}_\ell^{\mathcal{D}'} \mid D'_i = d_i, 1 \leq i \leq \ell\right\} \\
&\quad \times \Pr\left\{\hat{D}_i = d_i, 1 \leq i \leq n \mid \sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \\
&= \binom{n}{\ell}^{-1} \sum_{\substack{\vec{d}_\ell \text{ s.t.} \\ \sum_{i=1}^{\ell} d_i = \ell}} \Pr\left\{\mathcal{B}_\ell^{\mathcal{D}'} \mid D'_i = d_i, 1 \leq i \leq \ell\right\} \Pr\left\{D'_i = d_i, 1 \leq i \leq \ell\right\} \\
&= \binom{n}{\ell}^{-1} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\}. \tag{3.14}
\end{aligned}$$

Substituting (3.14) into (3.11), we obtain

$$\Pr\left\{\mathcal{C}_1^{\hat{D}} = [\ell]\right\} = \binom{n}{\ell}^{-1} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}.$$

It follows by the same argument and from the exchangeability of the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ that, for any $C \subseteq [n]$ such that $1 \in C$ and $|C| = \ell$, we have

$$\Pr\left\{\mathcal{C}_1^{\hat{D}} = C\right\} = \binom{n}{\ell}^{-1} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}.$$

So

$$\begin{aligned}
\Pr\left\{C_1^{\hat{D}}(n) = \ell\right\} &= \binom{n-1}{\ell-1} \binom{n}{\ell}^{-1} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\} \\
&= \frac{\ell}{n} \Pr\{\mathcal{B}_\ell^{\mathcal{D}'}\} \Pr\left\{\sum_{i=1}^{\ell} \hat{D}_i = \ell\right\}
\end{aligned}$$

as desired. \square

We prove an extension of Theorem 2 in the following special case: Suppose that D_1, D_2, \dots is a sequence of i.i.d. non-negative integer valued random variables. For each $n \geq 1$, let $\hat{D}(n) = (\hat{D}_{1,n}, \dots, \hat{D}_{n,n})$ be a sequence of variables with joint distribution given by

$$\Pr\{\hat{D}_{i,n} = d_i \text{ for } i = 1, 2, \dots, n\} = \Pr\{D_i = d_i, \quad i = 1, 2, \dots, n \mid \sum_{i=1}^n D_i = n\}.$$

Let $\mathcal{C}_1^{\hat{D}(n)}$ denote the vertex set of the connected component in $G_n^{\hat{D}(n)} \equiv G(T_n^{\hat{D}(n)})$ which contains the vertex labelled 1. For $k > 1$, we define $\mathcal{C}_k^{\hat{D}(n)}$ recursively as follows: If $[n] \setminus (\mathcal{C}_1^{\hat{D}(n)} \cup \dots \cup \mathcal{C}_{k-1}^{\hat{D}(n)}) \neq \emptyset$, let $\mathcal{C}_k^{\hat{D}(n)}$ denote the vertex set of the connected component in $G_n^{\hat{D}(n)}$ which contains the smallest element of $[n] \setminus (\mathcal{C}_1^{\hat{D}(n)} \cup \dots \cup \mathcal{C}_{k-1}^{\hat{D}(n)})$; otherwise, set $\mathcal{C}_k^{\hat{D}(n)} = \emptyset$. For all $k \geq 1$, let $C_k^{\hat{D}(n)} = |\mathcal{C}_k^{\hat{D}(n)}|$, then we have

Theorem 3. Suppose $1 \leq k < n$ and $\ell_1, \ell_2, \dots, \ell_k$ are such that $1 \leq \ell_1 < n$, $1 \leq \ell_i < n - \ell_0 - \ell_1 - \dots - \ell_{i-1} \leq n$, for $i = 2, \dots, k$, and $\sum_{i=1}^k \ell_i \leq n$. Then we have

$$\begin{aligned} & \Pr\{C_1^{\hat{D}(n)} = \ell_1, \dots, C_k^{\hat{D}(n)} = \ell_k\} = \\ &= \left[\prod_{i=1}^k \left(\frac{\ell_i}{n - t_{i-1}} \right) \Pr\{\mathcal{B}_{\ell_i}^{\hat{D}(\ell_i)}\} \right] \times \Pr\left\{ \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, \quad 1 \leq i \leq k \right\}, \end{aligned}$$

where $t_0 = 0$ and $t_i \equiv \ell_1 + \dots + \ell_i$, $i = 1, 2, \dots, k$.

Proof. For any integers $0 < m_1 < m_2$, let $[m_1, m_2]$ denote the set of integers $\{m_1, m_1 + 1, \dots, m_2\}$. Then we have

$$\begin{aligned} & \Pr\{C_1^{\hat{D}(n)} = \ell_1, \dots, C_k^{\hat{D}(n)} = \ell_k\} = \\ &= \prod_{i=1}^k \binom{n - t_{i-1} - 1}{\ell_i - 1} \times \Pr\{C_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k\}. \quad (3.15) \end{aligned}$$

Since

$$\{C_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i], \quad 1 \leq i \leq k\} \subseteq \left\{ \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, \quad 1 \leq i \leq k \right\}$$

we have

$$\begin{aligned}
& \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \right\} \\
&= \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \left| \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right. \right\} \\
&\quad \times \Pr \left\{ \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right\}. \tag{3.16}
\end{aligned}$$

Let $\mathcal{A} = \{\vec{d} = (d_1, \dots, d_n) : \sum_{j=t_{i-1}+1}^{t_i} d_j = \ell_i, 1 \leq i \leq k \text{ and } \sum_{j=1}^n d_j = n\}$, then

$$\begin{aligned}
& \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \left| \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right. \right\} \\
&= \sum_{\vec{d} \in \mathcal{A}} \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \left| \hat{D}_{j,n} = d_j, 1 \leq j \leq n \right. \right\} \\
&\quad \times \Pr \left\{ \hat{D}_{j,n} = d_j, 1 \leq j \leq n \left| \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right. \right\}. \tag{3.17}
\end{aligned}$$

Now for any $1 \leq i \leq k$, let $M(d_{t_{i-1}+1}, d_{t_{i-1}+2}, \dots, d_{t_i})$ denote the number of mappings of $[t_{i-1} + 1, t_i]$ into $[t_{i-1} + 1, t_i]$ such that the corresponding directed graph on the vertex set $\{t_{i-1} + 1, \dots, t_i\}$ is connected and the in-degree of each vertex $t_{i-1} + 1 \leq m \leq t_i$ is d_m . Then for any $\vec{d} \in \mathcal{A}$, we have (as in the proof of Theorem 2)

$$\begin{aligned}
& \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \left| \hat{D}_{j,n} = d_j, 1 \leq j \leq n \right. \right\} \\
&= \prod_{i=1}^k \binom{n - t_{i-1}}{\ell_i}^{-1} \times \prod_{i=1}^k \frac{M(d_{t_{i-1}+1}, \dots, d_{t_i})(d_{t_{i-1}+1})! \cdots d_{t_i}!}{\ell_i!} \\
&= \prod_{i=1}^k \binom{n - t_{i-1}}{\ell_i}^{-1} \times \prod_{i=1}^k \Pr \left\{ \mathcal{B}_{\ell_i}^{\hat{D}(\ell_i)} \left| \hat{D}_{j,\ell_i} = d_{t_{i-1}+j} \text{ for } 1 \leq j \leq \ell_i \right. \right\}. \tag{3.18}
\end{aligned}$$

The last equality follows by ‘re-labelling’ , for each $1 \leq i \leq k$, the vertices $t_{i-1} + 1, \dots, t_i$ by $1, 2, \dots, \ell_i$.

Next we note that since the variables D_1, D_2, \dots, D_n are independent and identically distributed, we have for every $\vec{d} \in \mathcal{A}$

$$\begin{aligned} & \Pr \left\{ \hat{D}_{j,n} = d_j, 1 \leq j \leq n \middle| \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right\} \\ &= \prod_{i=1}^k \frac{\Pr\{D_j = d_j, t_{i-1} + 1 \leq j \leq t_i\}}{\Pr\{\sum_{j=t_{i-1}+1}^{t_i} D_j = \ell_i\}} \times \frac{\Pr\{D_j = d_j, t_k + 1 \leq j \leq n\}}{\Pr\{\sum_{j=t_k+1}^n D_j = n - t_k\}} \\ &= \prod_{i=1}^k \Pr\left\{ \hat{D}_{j,\ell_i} = d_{t_{i-1}+j}, 1 \leq j \leq \ell_i \right\} \times \frac{\Pr\{D_j = d_j, t_k + 1 \leq j \leq n\}}{\Pr\{\sum_{j=t_k+1}^n D_j = n - t_k\}} \end{aligned} \quad (3.19)$$

where we replace the the last factor in the product by 1 if $t_k = n$. Substituting (3.18) and (3.19) into (3.17) and summing over all $\vec{d} \in \mathcal{A}$, we obtain

$$\begin{aligned} & \Pr \left\{ \mathcal{C}_i^{\hat{D}(n)} = [t_{i-1} + 1, t_i] \text{ for } i = 1, 2, \dots, k \middle| \sum_{j=t_{i-1}+1}^{t_i} \hat{D}_{j,n} = \ell_i, 1 \leq i \leq k \right\} \\ &= \prod_{i=1}^k \binom{n - t_{i-1}}{\ell_i}^{-1} \times \prod_{i=1}^k \Pr \left\{ \mathcal{B}_{\ell_i}^{\hat{D}(\ell_i)} \right\}. \end{aligned} \quad (3.20)$$

Finally, substituting (3.20) into (3.16) and then substituting (3.16) into (3.15), we obtain the result. \square

In the general case where the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ are exchangeable and $\sum_{i=1}^n \hat{D}_i = n$, an analogue of Theorem 3 can be proved. However, the result for the special case of Theorem 3 is easier because we are able to exploit the fact that the variables D_1, D_2, \dots are i.i.d. to obtain the product in (3.19) and hence the product in (3.20). Theorems 1,2, and 3 and their corollaries illustrate how the distributions of random mapping statistics for $T_n^{\hat{D}}$ can be computed in terms of the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$. Similar results for local properties of $T_n^{\hat{D}}$ such as the number of predecessors and the number of successors of a given vertex (or vertices) are obtained in a companion paper [23].

4 Examples

We consider two special examples which correspond, respectively, to a random mapping with ‘preferential attachment’ and a random mapping with ‘anti-preferential attachment’.

A Preferential Attachment Model

In this section we investigate $T_n^\rho : [n] \rightarrow [n]$, a random mapping with ‘preferential attachment’, where $\rho > 0$ is a fixed parameter. For $1 \leq k \leq n$, we define $T_n^\rho(k) = X_k^{(\rho, n)}$ where $X_1^{(\rho, n)}, X_2^{(\rho, n)}, \dots, X_n^{(\rho, n)}$ is a sequence of random variables whose distributions depend on the evolution of an urn scheme. The distribution of each $X_k^{(\rho, n)}$ is determined by a (random) n -tuple of non-negative weights $\vec{a}(k) = (a_1(k), a_2(k), \dots, a_n(k))$ where, for $1 \leq j \leq n$, $a_j(k)$ is the ‘weight’ of the j^{th} urn at the *start* of the k^{th} round of the urn scheme. Specifically, given $\vec{a}(k) = \vec{a} = (a_1, \dots, a_n)$, we define

$$\Pr \left\{ X_k^{(\rho, n)} = j \mid \vec{a}(k) = \vec{a} \right\} = \frac{a_j}{\sum_{i=1}^n a_i}.$$

The random weight vectors $\vec{a}(1), \vec{a}(2), \dots, \vec{a}(n)$ associated with the urn scheme are determined recursively. For $k = 1$, we set $a_1(1) = a_2(1) = \dots = a_n(1) = \rho > 0$. For $k > 1$, $\vec{a}(k)$ depends on both $\vec{a}(k-1)$ and the value of $X_{k-1}^{(\rho, n)}$ as follows: Given that $X_{k-1}^{(\rho, n)} = j$, we set $a_j(k) = a_j(k-1) + 1$ and for all other $i \neq j$, we set $a_i(k) = a_i(k-1)$ (i.e. if $X_{k-1}^{(\rho, n)} = j$ then a ‘ball, with weight 1 is added to the j^{th} urn’).

The random mapping T_n^ρ as defined above is a preferential attachment model in the following sense. Since, for $1 \leq k \leq n$, $T_n^\rho(k) = X_k^{(\rho, n)}$, and since the (conditional) distribution of $X_k^{(\rho, n)}$ depends on the state of the urn scheme at the start of round k , it is clear that vertex k is more likely to be mapped to vertex j if the weight $a_j(k)$ is (relatively) large, i.e. if several of the vertices $1, 2, \dots, k-1$ have already been mapped to vertex j . In the following proposition we establish the distribution of T_n^ρ .

Theorem 4. *Suppose that $D_1^\rho, D_2^\rho, \dots$ are i.i.d. random variables with a generalized negative binomial distribution given by*

$$\Pr\{D_1^\rho = k\} = \frac{\Gamma(k + \rho)}{k! \Gamma(\rho)} \left(\frac{\rho}{1 + \rho} \right)^\rho \left(\frac{1}{1 + \rho} \right)^k \quad \text{for } k = 0, 1, \dots,$$

where $\rho > 0$ is a fixed parameter.

For $n \geq 1$, let $\hat{D}(\rho, n) = (\hat{D}_{1,n}^\rho, \hat{D}_{2,n}^\rho, \dots, \hat{D}_{n,n}^\rho)$ be a sequence of variables with joint distribution given by

$$\Pr\{\hat{D}_{i,n}^\rho = d_i, 1 \leq i \leq n\} = \Pr\left\{D_i^\rho = d_i, 1 \leq i \leq n \middle| \sum_{i=1}^n D_i^\rho = n\right\}.$$

Then for every $n \geq 1$, the random mappings $T_n^\rho : [n] \rightarrow [n]$ and $T_n^{\hat{D}(\rho, n)} : [n] \rightarrow [n]$ have the same distribution.

Proof. To prove the result it is enough to show that for any $n \geq 1$ and any $f \in \mathcal{M}_n$

$$\Pr\{T_n^\rho = f\} = \Pr\{T_n^{\hat{D}(\rho, n)} = f\}.$$

Suppose that $f \in \mathcal{M}_n$ and that $\vec{d}(f) = (d_1, d_2, \dots, d_n)$. It is straightforward to check that

$$\begin{aligned} \Pr\{\hat{D}_{i,n}^\rho = d_i, 1 \leq i \leq n\} &= \Pr\left\{D_i^\rho = d_i, 1 \leq i \leq n \middle| \sum_{i=1}^n D_i^\rho = n\right\} \\ &= \prod_{i=1}^n \frac{\Gamma(d_i + \rho)}{(d_i)! \Gamma(\rho)} \times \frac{n! \Gamma(n\rho)}{\Gamma(n + n\rho)} \end{aligned}$$

since the distribution of $\sum_{i=1}^n D_i^\rho$ is given by

$$\Pr\left\{\sum_{i=1}^n D_i^\rho = k\right\} = \frac{\Gamma(k + n\rho)}{k! \Gamma(n\rho)} \left(\frac{\rho}{1 + \rho}\right)^{n\rho} \left(\frac{1}{1 + \rho}\right)^k \quad \text{for } k = 0, 1, \dots \quad (4.1)$$

It follows from the definition of $T_n^{\hat{D}(\rho, n)}$ that

$$\begin{aligned} \Pr\{T_n^{\hat{D}(\rho, n)} = f\} &= \frac{d_1! \cdots d_n!}{n!} \times \prod_{i=1}^n \frac{\Gamma(d_i + \rho)}{(d_i)! \Gamma(\rho)} \times \frac{n! \Gamma(n\rho)}{\Gamma(n + n\rho)} \\ &= \frac{\prod_{i=1}^n (d_i + \rho - 1)_{d_i}}{(n\rho + n - 1)_n} \end{aligned}$$

where $(m)_k \equiv m(m-1) \cdots (m-k+1)$ (and $(m)_0 = 1$).

Next, observe that if $T_n^\rho = f$ and $\vec{d}(f) = (d_1, d_2, \dots, d_n)$, then for each $1 \leq i \leq n$, d_i ‘balls’ of weight 1 are added to i^{th} urn during the evolution of

the urn scheme described above. So it follows from the definition of T_n^ρ in terms of the urn scheme, that

$$\Pr\{T_n^\rho = f\} = \frac{\prod_{i=1}^n (d_i + \rho - 1)_{d_i}}{(n\rho + n - 1)_n}.$$

The result follows since $n \geq 1$ and $f \in \mathcal{M}_n$ were arbitrary. \square

It is clear from Theorem 4 that the order in which a realisation of T_n^ρ is sequentially constructed does not matter. In particular, suppose that i_1, i_2, \dots, i_n is a permutation of $[n]$ and for $1 \leq k \leq n$, let $\tilde{T}_n^\rho(i_k) \equiv X_n^{(\rho, n)}$ where the variables $X_1^{(\rho, n)}, X_2^{(\rho, n)}, \dots, X_n^{(\rho, n)}$ are as defined above. Then it follows from the proof of Theorem 4 that $\tilde{T}_n^\rho \stackrel{d}{\sim} T_n^{\hat{D}(\rho, n)} \stackrel{d}{\sim} T_n^\rho$.

Since $T_n^\rho \stackrel{d}{\sim} T_n^{\hat{D}(\rho, n)}$, we can investigate the structure of $G_n^\rho \equiv G(T_n^\rho)$ by considering the structure of $G_n^{\hat{D}(\rho, n)}$. In this paper, in order to illustrate the general method, we derive both exact and asymptotic results for X_n^ρ , the number of cyclic vertices in G_n^ρ , and $C_1^\rho(n)$, the size of the component in G_n^ρ which contains vertex 1. Further results concerning the structure of G_n^ρ can be found in [24] where we also consider the structure of G_n^ρ when $\rho = \rho(n)$ is a function of n .

By Theorem 1 and Theorem 4 we have, for $1 \leq k < n$,

$$\Pr\{X_n^\rho = k\} = \Pr\{X_n^{\hat{D}(\rho, n)} = k\} = \frac{k}{n-k} E\left((\hat{D}_{1,n}^\rho - 1)\hat{D}_{1,n}^\rho \hat{D}_{2,n}^\rho \cdots \hat{D}_{k,n}^\rho\right). \quad (4.2)$$

Since

$$E\left((\hat{D}_{1,n}^\rho - 1)\hat{D}_{1,n}^\rho \hat{D}_{2,n}^\rho \cdots \hat{D}_{k,n}^\rho\right) = E\left((D_1^\rho - 1)D_1^\rho D_2^\rho \cdots D_k^\rho \mid \sum_{i=1}^n D_i^\rho = n\right)$$

we have

$$\begin{aligned} & E\left((\hat{D}_{1,n}^\rho - 1)\hat{D}_{1,n}^\rho \hat{D}_{2,n}^\rho \cdots \hat{D}_{k,n}^\rho\right) = \\ &= \frac{\text{coefficient of } s^n \text{ in } E\left((D_1^\rho - 1)D_1^\rho s^{D_1^\rho} D_2^\rho s^{D_2^\rho} \cdots D_k^\rho s^{D_k^\rho} s^{D_{k+1}^\rho} \cdots s^{D_n^\rho}\right)}{\text{coefficient of } s^n \text{ in } E\left(s^{D_1^\rho} \cdots s^{D_n^\rho}\right)} \\ &= \frac{\text{coefficient of } s^n \text{ in } E\left((D_1^\rho - 1)D_1^\rho s^{D_1^\rho}\right) \left(E\left(D_1^\rho s^{D_1^\rho}\right)\right)^{k-1} \left(E\left(s^{D_1^\rho}\right)\right)^{n-k}}{\text{coefficient of } s^n \text{ in } \left(E\left(s^{D_1^\rho}\right)\right)^n}. \end{aligned} \quad (4.3)$$

The last equality holds since the variables $D_1^\rho, D_2^\rho, \dots, D_n^\rho$ are independent and identically distributed. Since the identity

$$\frac{1}{(1-u)^\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} u^k$$

holds for all $\alpha > 0$ and $|u| < 1$, we have

$$\begin{aligned} E\left(s^{D_1^\rho}\right) &= \left(\frac{\rho}{1+\rho-s}\right)^\rho, \quad E\left(D_1^\rho s^{D_1^\rho}\right) = s \left(\frac{\rho}{1+\rho-s}\right)^{\rho+1}, \quad \text{and} \\ E\left((D_1^\rho - 1)D_1^\rho s^{D_1^\rho}\right) &= \left(\frac{1+\rho}{\rho}\right) s^2 \left(\frac{\rho}{1+\rho-s}\right)^{\rho+2}. \end{aligned} \quad (4.4)$$

It follows from (4.2)-(4.4) and routine calculations that for $1 \leq k < n$,

$$\begin{aligned} \Pr\{X_n^\rho = k\} &= \frac{k}{n-k} \rho^k (1+\rho) \frac{\Gamma(n\rho)n!}{(n-k-1)!\Gamma(n\rho+k+1)} \\ &= k\rho^k (1+\rho) \frac{(n)_k}{(n\rho+k)_{k+1}} \end{aligned} \quad (4.5)$$

and for $k = n$, we have

$$\Pr\{X_n^\rho = n\} = \Pr\{\hat{D}_{i,n}^\rho = 1, 1 \leq i \leq n\} = \frac{\rho^n n! \Gamma(n\rho)}{\Gamma(n+n\rho)}. \quad (4.6)$$

Next, we consider the limiting distribution of X_n^ρ : Fix $0 < x < \infty$ and suppose that $k = \lfloor x\sqrt{n} \rfloor$, then we have

$$\begin{aligned} \Pr\{X_n^\rho = k\} &= k\rho^k (1+\rho) \frac{(n)_k}{(n\rho+k)_{k+1}} \\ &= \frac{k}{n} \left(\frac{1+\rho}{\rho}\right) \frac{(1-\frac{1}{n}) \cdots (1-\frac{k-1}{n})}{(1+\frac{k}{n\rho}) \cdots (1+\frac{1}{n\rho})} \\ &\sim \left(\frac{1+\rho}{\rho}\right) x \exp\left(-\frac{(1+\rho)x^2}{2\rho}\right) \frac{1}{\sqrt{n}}. \end{aligned} \quad (4.7)$$

Hence X_n^ρ/\sqrt{n} converges in distribution to a variable \tilde{X}_ρ with density

$$f_{\tilde{X}_\rho}(x) = \left(\frac{1+\rho}{\rho}\right) x \exp\left(-\frac{(1+\rho)x^2}{2\rho}\right) \text{ for } x \geq 0, \quad (4.8)$$

and hence $E(X_n^\rho) \sim \sqrt{\frac{\rho\pi n}{2(1+\rho)}}$. Let N_n^ρ denote the number of components in G_n^ρ , then, using standard arguments (see [41]), it follows from Corollary 3 and Theorem 4 that $(N_n^\rho - \frac{1}{2} \log n) / \sqrt{\frac{1}{2} \log n}$ converges in distribution to the standard $N(0, 1)$ distribution.

We apply Theorem 3 and Theorem 4 to obtain the distribution for $C_1^\rho(n)$:

$$\begin{aligned} \Pr\{C_1^\rho(n) = \ell\} &= \Pr\{C_1^{\hat{D}(\rho, n)} = \ell\} \\ &= \frac{\ell}{n} \Pr\{\mathcal{B}_\ell^{\hat{D}(\rho, \ell)}\} \Pr\left\{\sum_{i=1}^{\ell} D_i^\rho \middle| \sum_{i=1}^n D_i^\rho = n\right\} \end{aligned} \quad (4.9)$$

for $1 \leq \ell \leq n$. To obtain a local limit theorem for the distribution of $C_1^\rho(n)$, fix $0 < x < \infty$ and suppose that $\ell = \lfloor xn \rfloor$. Then it follows from Theorem 4, Corollary 2, (4.5), and (4.6) that

$$\begin{aligned} \Pr\{\mathcal{B}_\ell^{\hat{D}(\rho, \ell)}\} &= \sum_{k=1}^{\ell} \frac{1}{k} \Pr\{X_\ell^{\hat{D}(\rho, \ell)} = k\} = \sum_{k=1}^{\ell-1} \rho^k (1+\rho) \frac{(\ell)_k}{(\ell\rho + k)_{k+1}} + \frac{\rho^n \ell! \Gamma(\ell\rho)}{\Gamma(\ell + \ell\rho)} \\ &\sim \frac{1}{\sqrt{\ell}} \int_0^\infty \left(\frac{1+\rho}{\rho}\right) \exp\left(-\frac{(1+\rho)x^2}{2\rho}\right) dx = \sqrt{\frac{1+\rho}{\rho}} \cdot \sqrt{\frac{\pi}{2\ell}}. \end{aligned} \quad (4.10)$$

Also, since the variables $D_1^\rho, D_2^\rho, \dots$ are independent, we have

$$\begin{aligned} \Pr\left\{\sum_{i=1}^{\ell} D_i^\rho \middle| \sum_{i=1}^n D_i^\rho = n\right\} &= \frac{\Pr\{\sum_{i=1}^{\ell} D_i^\rho\} \Pr\{\sum_{i=\ell+1}^n D_i^\rho\}}{\Pr\{\sum_{i=1}^n D_i^\rho\}} \\ &= \binom{n}{\ell} \frac{\Gamma(\ell(1+\rho)) \Gamma((n-\ell)(1+\rho)) \Gamma(n\rho)}{\Gamma(\ell\rho) \Gamma((n-\ell)\rho) \Gamma(n(1+\rho))} \\ &\sim \sqrt{\frac{\rho}{1+\rho}} \sqrt{\frac{n}{2\pi\ell(n-\ell)}}. \end{aligned} \quad (4.11)$$

Substituting (4.10) and (4.11) into (4.9), we obtain

$$\Pr\{C_1^\rho(n) = \ell\} = \Pr\{C_1^\rho(n) = \lfloor xn \rfloor\} \sim \frac{1}{2n\sqrt{1-x}}. \quad (4.12)$$

So, $\frac{C_1^\rho(n)}{n}$ converges in distribution to Z_1 as $n \rightarrow \infty$, where Z_1 has density $f_{Z_1}(u) = \theta(1-u)^{\theta-1}$ on the interval $(0,1)$ with $\theta = 1/2$. It follows from Theorem 3 and similar calculations that for any integer $t \geq 1$ and constants $0 < a_i < b_i < 1$, where $1 \leq i \leq t$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr \left\{ a_i < \frac{C_i^{\hat{D}(\rho,n)}}{n - C_1^{\hat{D}(\rho,n)} - \dots - C_{i-1}^{\hat{D}(\rho,n)}} < b_i, \quad 1 \leq i \leq t \right\} \\ = \prod_{i=1}^t \int_{a_i}^{b_i} \frac{1}{2\sqrt{1-x}} dx. \end{aligned}$$

Hence, it follows from standard arguments (see, for example [22]) that:

Theorem 5. *The joint distribution of the normalized order statistics for the component sizes in G_n^ρ converges to the Poisson-Dirichlet (1/2) distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \text{ for every } i \geq 1\}$.*

An Anti-Preferential Attachment Model

In this section we define $T_n^m : [n] \rightarrow [n]$, a random mapping with ‘anti-preferential attachment’, where $m \geq 1$ is a fixed interger parameter. For $1 \leq k \leq n$, we define $T_n^m(k) = Y_k^{(m,n)}$ where, as in the definition of T_n^ρ , the variables $Y_1^{(m,n)}, Y_2^{(m,n)}, \dots, Y_n^{(m,n)}$ depend on the evolution of an urn scheme. The distribution of each variable $Y_k^{(m,n)}$ is determined by a (random) n -tuple of non-negative weights $\vec{b}(k) = (b_1(k), b_2(k), \dots, b_n(k))$ where, for $1 \leq j \leq n$, $b_j(k)$ is the number of balls in the j^{th} urn at the start of the k^{th} round of the urn scheme. Specifically, given $\vec{b}(k) = \vec{b} = (b_1, \dots, b_n)$, we define

$$\Pr \left\{ Y_k^{(m,n)} = j \mid \vec{b}(k) = \vec{b} \right\} = \frac{b_j}{\sum_{i=1}^n b_i}.$$

The random weight vectors $\vec{b}(1), \vec{b}(2), \dots, \vec{b}(n)$ associated with the urn scheme are determined recursively. For $k = 1$, we set $b_1(1) = b_2(1) = \dots = b_n(1) = m$. For $k > 1$, $\vec{b}(k)$ depends on both $\vec{b}(k-1)$ and the value of $Y_{k-1}^{(m,n)}$ as follows: Given that $Y_{k-1}^{(m,n)} = j$, we set $b_j(k) = b_j(k-1) - 1$ and for all other $i \neq j$, we set $b_i(k) = b_i(k-1)$ (i.e. if $Y_{k-1}^{(m,n)} = j$ then a ball is removed from the j^{th} urn).

The random mapping T_n^m as defined above is an anti-preferential attachment model in the following sense. Since, for $1 \leq k \leq n$, we have $T_n^m(k) = Y_k^{(m,n)}$, and since the (conditional) distribution of $Y_k^{(m,n)}$ depends on the state of the urn scheme at the start of round k , it is clear that vertex k is less likely to ‘choose’ vertex j if the weight $b_j(k)$ is (relatively) small, i.e. if several of the vertices $1, 2, \dots, k-1$ have already been mapped to vertex j . It is also clear from the definition of T_n^m that the in-degree of any vertex in the random digraph $G_n^m \equiv G(T_n^m)$ is at most m and in the case $m = 1$, T_n^1 is a (uniform) random permutation. In the following proposition we determine the distribution of T_n^m .

Theorem 6. *Suppose that D_1^m, D_2^m, \dots are i.i.d. $\text{Bin}(m, \frac{1}{m})$ variables where $m \geq 1$ is a fixed integer parameter. Let $\hat{D}(m, n) = (\hat{D}_{1,n}^m, \hat{D}_{2,n}^m, \dots, \hat{D}_{n,n}^m)$ be a sequence of variables with joint distribution given by*

$$\Pr\{\hat{D}_{i,n}^m = d_i, 1 \leq i \leq n\} = \Pr\left\{D_i^m = d_i, 1 \leq i \leq n \mid \sum_{i=1}^n D_i^m = n\right\}.$$

Then the random mappings T_n^m and $T_n^{\hat{D}(m,n)}$ have the same distribution.

Proof. To prove the result it is enough to show that for any $n \geq 1$ and any $f \in \mathcal{M}_n$

$$\Pr\{T_n^m = f\} = \Pr\{T_n^{\hat{D}(m,n)} = f\}.$$

Suppose that $f \in \mathcal{M}_n$ and that $\vec{d}(f) = (d_1, d_2, \dots, d_n)$. It is straightforward to check that

$$\begin{aligned} \Pr\{\hat{D}_{i,n}^m = d_i, 1 \leq i \leq n\} &= \Pr\left\{D_i^m = d_i, 1 \leq i \leq n \mid \sum_{i=1}^n D_i^m = n\right\} \\ &= \frac{\prod_{i=1}^n \binom{m}{d_i}}{\binom{nm}{n}}, \end{aligned}$$

and hence, from the definition of $T_n^{\hat{D}(m,n)}$, that

$$\Pr\{T_n^{\hat{D}(m,n)} = f\} = \frac{d_1! \cdots d_n!}{n!} \times \frac{\prod_{i=1}^n \binom{m}{d_i}}{\binom{nm}{n}}$$

$$= \frac{\prod_{i=1}^n (m)_{d_i}}{(nm)_n}.$$

On the other hand, $T_n^m = f$ with $\vec{d}(f) = (d_1, d_2, \dots, d_n)$ if and only if for each $1 \leq i \leq n$, d_i balls are removed, in a certain order, from the i^{th} urn during the evolution of the urn model described above. So it follows from the definition of T_n^m in terms of the urn scheme, that

$$\Pr\{T_n^m = f\} = \frac{\prod_{i=1}^n (m)_{d_i}}{(nm)_n}.$$

The result follows since $n \geq 1$ and $f \in \mathcal{M}_n$ were arbitrary. \square

Again, it is clear from Theorem 6 that the order in which a realisation of T_n^m is sequentially constructed does not matter. In particular, suppose that i_1, i_2, \dots, i_n is a permutation of $[n]$ and for $1 \leq k \leq n$, let $\tilde{T}_n^m(i_k) \equiv Y_n^{(m,n)}$ where the variables $Y_1^{(m,n)}, Y_2^{(m,n)}, \dots, Y_n^{(m,n)}$ are as defined above. Then it follows from the proof of Theorem 6 that $\tilde{T}_n^m \stackrel{d}{\sim} T_n^{\hat{D}(m,n)} \stackrel{d}{\sim} T_n^m$.

We apply Theorem 6 to investigate the distributions of the number of cyclic vertices in $G_n^m \equiv G(T_n^m)$ and the size of a typical component in $G(T_n^m)$. Let X_n^m denote the number of cyclic vertices in the random digraph $G_n^m \equiv G(T_n^m)$ and let $C_1^m(n)$ denote the size of the component in G_n^m which contains the vertex 1. Since $T_n^m \stackrel{d}{\sim} T_n^{\hat{D}(m,n)}$, we have $G(T_n^m) \stackrel{d}{\sim} G_n^{\hat{D}(m,n)}$ and $X_n^m \stackrel{d}{\sim} X_n^{\hat{D}(m,n)}$. So, it follows from Theorem 1 and Theorem 6 (and its proof) that for $m \geq 2$ and $1 \leq k < n$, we have

$$\Pr\{X_n^m = k\} = \Pr\{X_n^{\hat{D}(m,n)} = k\} = \frac{k}{n-k} E((\hat{D}_{1,n}^m - 1) \hat{D}_{1,n}^m \hat{D}_{2,n}^m \cdots \hat{D}_{k,n}^m)$$

$$\begin{aligned} &= \frac{k}{n-k} \sum_{\vec{d} \text{ s.t. } \sum_{i=1}^n d_i = n} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_n}}{\binom{nm}{n}} \\ &= \frac{k}{n-k} \sum_{t=k+1}^{\min(n, km)} \sum_{\substack{\vec{d} \text{ s.t. } \sum_{i=1}^k d_i = t \\ \text{and } \sum_{i=1}^n d_i = n}} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_n}}{\binom{nm}{n}} \end{aligned}$$

$$\begin{aligned}
&= \frac{k}{n-k} \sum_{t=k+1}^{\min(n, km)} \sum_{\vec{d} \text{ s.t. } \sum_{i=1}^k d_i = t} (d_1 - 1) d_1 d_2 \cdots d_k \times \frac{\binom{m}{d_1} \cdots \binom{m}{d_k} \binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \sum_{t=k+1}^{\min(n, km)} \sum_{\vec{d} \text{ s.t. } \sum_{i=1}^k d_i = t} \frac{\binom{m-2}{d_1-2} \binom{m-1}{d_2-1} \cdots \binom{m-1}{d_k-1} \binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \sum_{t=k+1}^{\min(n, km)} \frac{\binom{km-k-1}{t-k-1} \binom{nm-km}{n-t}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \frac{\binom{nm-k-1}{n-k-1}}{\binom{nm}{n}}.
\end{aligned}$$

In the summations above the sum is always taken over those degree sequences for which the binomial coefficients are defined. We also adopt the formal convention that $\binom{0}{0} = 1$. Finally, for $k = n$ and $m \geq 2$, we obtain

$$\Pr\{X_n^m = n\} = \Pr\{\hat{D}_{i,n}^m = 1, 1 \leq i \leq n\} = \frac{m^n}{\binom{nm}{n}},$$

and when $m = 1$, we have $X_n^1 \equiv n$.

To obtain a local limit theorem for X_n^m , fix $0 < x < \infty$ and suppose that $k = \lfloor x\sqrt{n} \rfloor$. Then we have

$$\begin{aligned}
\Pr\{X_n^m = k\} &= \frac{k}{n-k} m^k (m-1) \frac{\binom{nm-k-1}{n-k-1}}{\binom{nm}{n}} \\
&= \frac{k}{n-k} m^k (m-1) \frac{(n)_{k+1}}{(nm)_{k+1}} \\
&\sim \left(\frac{m-1}{m}\right) x \exp\left(\frac{-(m-1)x^2}{2m}\right) \frac{1}{\sqrt{n}}.
\end{aligned}$$

It follows that X_n^m/\sqrt{n} converges in distribution to a variable \tilde{X}_m with density

$$f_{\tilde{X}_m}(x) = \left(\frac{m-1}{m}\right) x \exp\left(\frac{-(m-1)x^2}{2m}\right) \quad \text{for } x \geq 0,$$

and hence $E(X_n^m) \sim \sqrt{\frac{mn\pi}{2(m-1)}}$. Let N_n^m denote the number of components in G_n^m . Then, by the same argument as used for N_n^ρ above, it follows that

$(N_n^m - \frac{1}{2} \log n) / \sqrt{\frac{1}{2} \log n}$ converges in distribution to a standard $N(0, 1)$ distribution.

We apply Theorem 3 and Theorem 6 to obtain the distribution for $C_1^m(n)$:

$$\begin{aligned} \Pr\{C_1^m(n) = \ell\} &= \Pr\{C_1^{\hat{D}(m,n)} = \ell\} \\ &= \frac{\ell}{n} \Pr\{\mathcal{B}_\ell^{\hat{D}(m,\ell)}\} \Pr\left\{\sum_{i=1}^{\ell} D_i^m \middle| \sum_{i=1}^n D_i^m = n\right\} \end{aligned} \quad (4.13)$$

for $1 \leq \ell \leq n$. To obtain a local limit theorem for the distribution of $C_1^m(n)$, fix $0 < x < \infty$ and suppose that $\ell = \lfloor xn \rfloor$. Then it follows from Theorem 6 and Corollary 2, that

$$\begin{aligned} \Pr\{\mathcal{B}_\ell^{\hat{D}(m,\ell)}\} &= \sum_{k=1}^{\ell} \frac{1}{k} \Pr\{X_\ell^{\hat{D}(m,\ell)} = k\} \\ &= \sum_{k=1}^{\ell-1} \frac{m^k (m-1)}{\ell-k} \frac{\binom{\ell m - k - 1}{\ell - k - 1}}{\binom{\ell m}{\ell}} + \frac{1}{\ell} \frac{m^\ell}{\binom{\ell m}{\ell}} \\ &\sim \frac{1}{\sqrt{\ell}} \int_0^\infty \left(\frac{m-1}{m}\right) \exp\left(-\frac{(m-1)x^2}{2m}\right) dx = \sqrt{\frac{m-1}{m}} \sqrt{\frac{\pi}{2\ell}}. \end{aligned} \quad (4.14)$$

Since D_1^m, D_2^m, \dots are i.i.d. $\text{Bin}(m, \frac{1}{m})$ variables, we also have

$$\Pr\left\{\sum_{i=1}^{\ell} D_i^m = \ell \middle| \sum_{i=1}^n D_i^m = n\right\} = \frac{\binom{m\ell}{\ell} \binom{mn-m\ell}{n-\ell}}{\binom{mn}{n}} \sim \frac{\sqrt{mn}}{\sqrt{2\pi\ell(m-1)(n-\ell)}}. \quad (4.15)$$

Substituting (4.14) and (4.15) into (4.13) we obtain

$$\Pr\{C_1^m(n) = \ell\} \sim \frac{1}{2n\sqrt{1-x}} \quad \text{as } n \rightarrow \infty. \quad (4.16)$$

It follows that as $n \rightarrow \infty$, $\frac{C_1^m(n)}{n}$ converges in distribution to Z_1 , where Z_1 has $f_{Z_1}(u) = \theta(1-u)^{\theta-1}$ on the interval $(0, 1)$ with parameter $\theta = 1/2$. Again, as in the case of the preferential attachment model, it is not difficult to extend (4.16) in order to show that:

Theorem 7. *The joint distribution of the normalized order statistics for the component sizes in G_n^m converges to the Poisson-Dirichlet $(1/2)$ distribution on the simplex $\nabla = \{\{x_i\} : \sum x_i \leq 1, x_i \geq x_{i+1} \geq 0 \text{ for every } i \geq 1\}$.*

5 Final Remarks

In this paper we have introduced a new random mapping model, $T_n^{\hat{D}}$, which is defined in terms of a collection of exchangeable, non-negative, integer-valued ‘in-degree’ variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ such that $\sum_{i=1}^n \hat{D}_i = n$. We note the classic model $T_{\mathbf{p}(n)}$ has the property that each vertex *independently* chooses its image under $T_{\mathbf{p}(n)}$. In $T_n^{\hat{D}}$ we have replaced this independence with exchangeability of the in-degrees in $G_n^{\hat{D}}$ (and in the examples with the ‘independence’ of random variables which are used to define the in-degrees).

We have shown that the joint distribution of the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ is the key to understanding both the local and global structure of $G_n^{\hat{D}}$, and that the distributions of several variables associated with the structure of $G_n^{\hat{D}}$ can be expressed in terms of expectations of various functions of $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$. In the special case where the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ have the same distribution as a collection of i.i.d. variables D_1, D_2, \dots, D_n *conditioned* on $\sum_{i=1}^n D_i = n$, we have shown that it is easy to apply our results to obtain exact and asymptotic distributions for the number of cyclic vertices, the number of components, and the size of a typical component in $G_n^{\hat{D}}$. Both $T_{\mathbf{p}(n)}$ and $T_n^{\hat{D}}$ share the property that these mappings restricted to the cyclical vertices are both uniform random permutations on the cyclical vertices. This means that given the distribution of the number of cyclic vertices, $X_n^{\hat{D}}$, (see Theorem 1) we can exploit known results for uniform random permutations to obtain results for $T_n^{\hat{D}}$, and it explains some similarities between results for both $T_{\mathbf{p}(n)}$ and $T_n^{\hat{D}}$.

The transparent relationship between the distribution of the variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ and the structure of random mapping $T_n^{\hat{D}}$ is the main advantage of this new model. It also provides the basis for testing and fitting random mapping models in various applications. For example, if $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ have the same distribution as n independent *Poisson*(1) variables, D_1, D_2, \dots, D_n , *conditioned* on $\sum_{i=1}^n D_i = n$, then $T_n^{\hat{D}}$ is a uniform random mapping and, for large n and $1 \leq i \leq n$, the marginal distribution of each \hat{D}_i is approximately *Poisson*(1). So vertex in-degree data, such as directed epidemic contacts, can be used to test whether the associated mapping is a realisation of a uniform random mapping model.

As another example, we mention the work of Arney and Bender on random mappings with constraints on coalescence [4]. Their work was motivated, in part, by the analysis of shift register data. In order to model a random

shift register they put a uniform measure on $\mathcal{M}_n^{\{0,1,2\}}$, the set of all mappings $f : [n] \rightarrow [n]$ such that, for every $1 \leq i \leq n$, $|f^{-1}(i)|$, the number of pre-images of i under f , equals 0, 1, or 2. So, if $f \in \mathcal{M}_n^{\{0,1,2\}}$, then every vertex in $G_n(f)$ has in-degree equal 0, 1, or 2. Arney and Bender observed that in some respects their model does not fit the shift register data. In particular, their model predicts $0.293n$ vertices with in-degree 0 whereas the average number of vertices with in-degree 0 in a random shift register is $n/4$. By using the model $T_n^{\hat{D}}$ instead, we can more successfully capture the local structure of the shift register data. Specifically, suppose that $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ have the same distribution as n independent $\text{Bin}(2, \frac{1}{2})$ variables, D_1, D_2, \dots, D_n , conditioned on $\sum_{i=1}^n D_i = n$. Then $\Pr\{\hat{D}_1 = 0\} = \frac{1}{4}(1 + \frac{1}{2n-2})^{-1}$ and the expected number of vertices with in-degree 0 in $T_n^{\hat{D}}$ is asymptotic to $\frac{n}{4}$. We note that, for example, the asymptotic distribution of the normalised typical component size is the *same* for both the Arney and Bender model and $T_n^{\hat{D}}$. So it is not surprising that Arney and Bender found that their model fit some other features of the shift register data quite well.

Finally, we mention that in applications of random mapping models in cryptography and in epidemic process modelling the distributions of the number of predecessors and of the number of successors of an arbitrary vertex or set of vertices are also of interest. We develop a calculus for computing these distributions based on the underlying variables $\hat{D}_1, \hat{D}_2, \dots, \hat{D}_n$ in a companion paper [23].

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